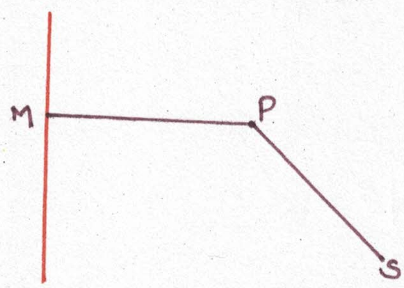

ELLPSE
STANDAL EQUCTION
Semi-Major Axis
Foci
Foci
Ellipse
Foci
ellipse

ELLIPSE

Defination

The locus of a pt. which moves in such a way that the ratio of its distance from a fixed pt. and a fixed line is always constant.

Value of constant lies in interval (0, 1)

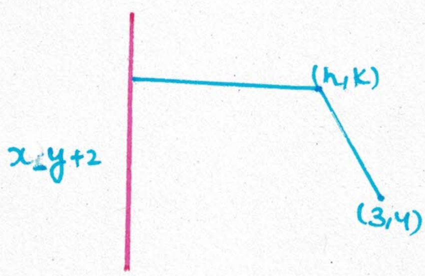


$\frac{SP}{PM} = e \qquad 0 < e < 1$

Hence, $SP = e\,PM$

Ques: Fixed pt. (3,4), Fixed line $x - y + 2 = 0$, $e = 1/2$

Sol:



$$(x-3)^2 + (y-4)^2 = \frac{1}{4}\,\frac{(x-y+2)^2}{2}$$

$$8\left[x^2+y^2-6x-8y+9+16\right] = x^2+y^2+4-2xy-4y+4x$$

$$\boxed{7x^2+7y^2-52x-60y+200+2xy=0}\ \text{\}Equation}$$

The second degree general equation

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

represents ellipse, when

$\Delta = abc + 2fgh - af^2 - bg^2 - ch^2 \neq 0$

$h^2 < ab$

Standard Equation of Ellipse

$S = (ae, 0) \qquad$ directrix $\Rightarrow x = \frac{a}{e} \qquad P(x, y)$

$SP = e\,PM$

$$(x-ae)^2 + (y-0)^2 = e^2\left(x - \frac{a}{e}\right)^2$$

$$\Rightarrow x^2 + a^2e^2 - 2aex + y^2 = e^2\left(x^2 + \frac{a^2}{e^2} - \frac{2xa}{e}\right)$$

$$\Rightarrow x^2 + a^2e^2 - 2aex + y^2 = e^2x^2 + a^2 - 2xae$$

$$\Rightarrow x^2(1-e^2) + y^2 = a^2(1-e^2)$$

$$\Rightarrow \frac{x^2}{a^2} + \frac{y^2}{a^2(1-e^2)} = 1$$

Let $b^2 = a^2(1-e^2)$

Then, $\boxed{\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1}$ } Standard equation of Horizontal Ellipse

$$e = \sqrt{\frac{a^2-b^2}{a^2}} = \boxed{\frac{\sqrt{a^2-b^2}}{a} = e}$$

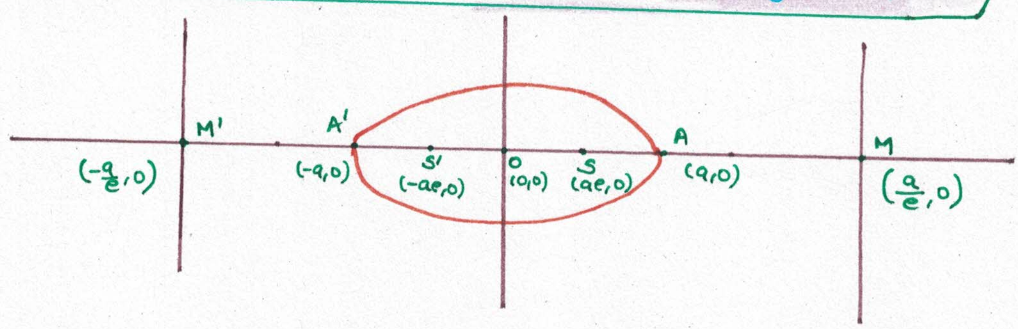
Vertical Ellipse:

$$\boxed{\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1 \qquad e = \sqrt{1 - \frac{b^2}{a^2}}}$$

eg. $\frac{x^2}{3} + \frac{y^2}{5} = 1$ Vertical ellipse $e = \sqrt{2/5}$

eg. $\frac{x^2}{16} + \frac{y^2}{9} = 1$ Horizontal ellipse $e = \sqrt{7}/4$

Derivation of Directrix and Focus of Ellipse



$AS = eAM$ —— ① $AS' = eAM'$ —— ②

$1+2 \Rightarrow$ $AS + AS' = e(AM + AM')$

$(OA - OS + OA + OS') = e(OM - OA + OM' + OA)$

$2OA = 2eOM$

$\frac{a}{e} = OM$

$$\boxed{M = \left(\frac{a}{e}, 0\right) \qquad M' = \left(-\frac{a}{e}, 0\right)}$$

$1-2 \Rightarrow \quad (OA - OS - OA - OS') = e(OM - OA - OM' - OA)$

$$2OS = 2eOA$$

$$OS = ae$$

$$\boxed{S = (ae, 0) \qquad S' = (-ae, 0)}$$

Some Terminologies

Major Axis (MA)

The line of symmetry which contains foci and vertices is called major axis.

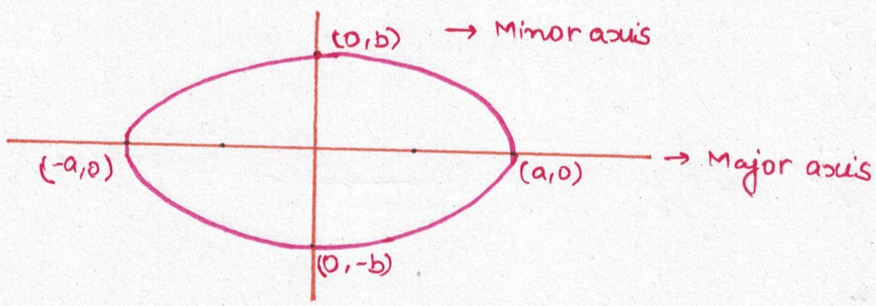
Minor Axis (ma)

The line perpendicular to major axis and passing through the centre of ellipse is called minor axis.

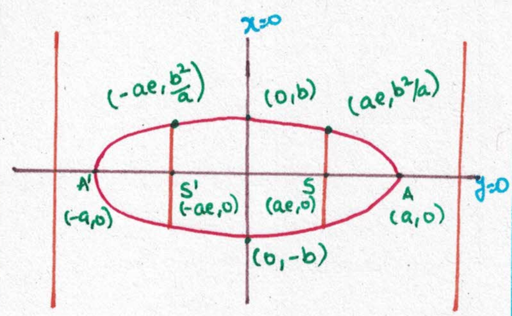
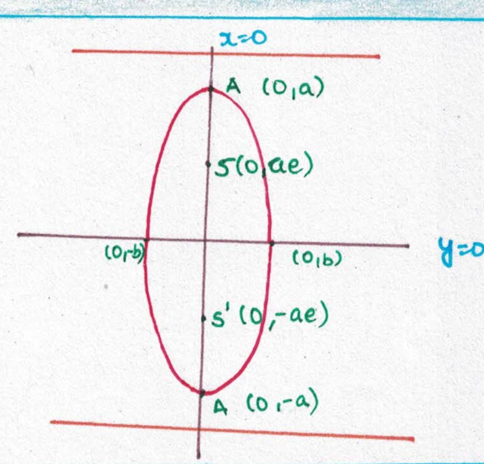
It is also known as shortest line of symmetry

Length of major axis = $2a$

Length of minor axis = $2b$

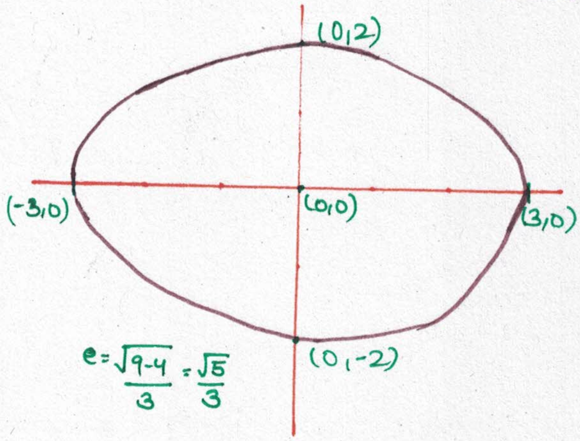


$$\boxed{\text{Length of latus rectum} = \frac{2b^2}{a}}$$

	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$	$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$
Graph		

vertex	$(\pm a, 0)$	$(0, \pm a)$
Focii	$(\pm ae, 0)$	$(0, \pm ae)$
Major axis	$y=0$	$x=0$
Minor axis	$x=0$	$y=0$
Directrix	$x = \pm a/e$	$y = \pm a/e$
End pt. of Minor axis	$(0, \pm b)$	$(\pm b, 0)$
Latus Rectum	$x = \pm ae$	$y = \pm ae$
Length of MA	$2a$	$2a$
Length of ma	$2b$	$2b$
LOLR	$2b^2/a$	$2b^2/a$
End pts. of LR	$(ae, b^2/a)$, $(ae, -b^2/a)$, $(-ae, b^2/a)$, $(-ae, -b^2/a)$	$(b^2/a, ae)$, $(-b^2/a, ae)$, $(b^2/a, -ae)$ $(-b^2/a, -ae)$
Eccentricity	$\sqrt{a^2-b^2}/a$	$\sqrt{a^2-b^2}/a$

Ques: $\frac{x^2}{9} + \frac{y^2}{4} = 1$



$e = \frac{\sqrt{9-4}}{3} = \frac{\sqrt{5}}{3}$

Sol: $a^2 = 9 \Rightarrow a = 3$, $b = 2$

1) $C = (0,0)$

2) $V = (\pm 3, 0)$

3) $S = (\pm\sqrt{5}, 0)$

4) major axis $\Rightarrow y = 0$

5) minor axis $\Rightarrow x = 0$

6) directrix $\Rightarrow \left(\pm \frac{a}{e}, 0\right) = \left(\pm \frac{9}{\sqrt{5}}, 0\right)$

7) LOMA $= 2a = 2 \times 3 = 6$

8) LOma $= 4$

9) LOLR $= \frac{2b^2}{a} = \frac{2\times 4}{3} = 8/3$

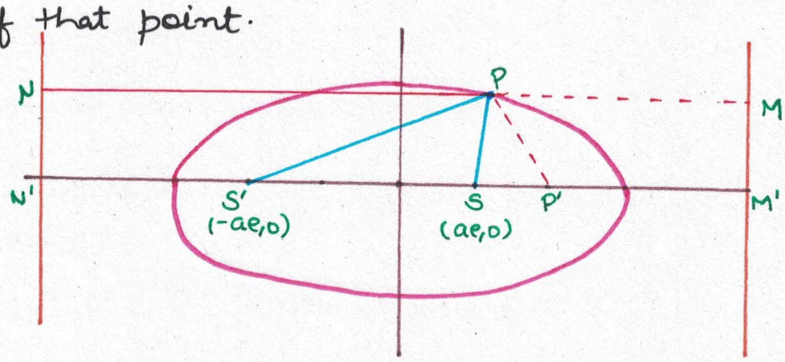
10) End pts. of LR $= \left(\sqrt{5}, \frac{4}{3}\right)$ $\left(-\sqrt{5}, \frac{4}{3}\right)$

11) $e = \sqrt{5}/3$

12) End pts. of Minor axis $= (0, \pm 2)$

Focal Length

The distance b/w focus and any pt. on the ellipse is known as focal length of that point.



Focal length PS :-

$$SP = e\,PM = e(P'M')$$

$$PS = e(OM' - OP')$$

$$PS = e\left(\frac{a}{e} - |x_1|\right)$$

$$\boxed{PS = a - e|x_1|}$$

Focal length PS' :-

$$PS' = e\,PN = e(P'N')$$

$$PS' = e(ON' + OP')$$

$$PS' = e\left(+\frac{a}{e} + |x_1|\right)$$

$$\boxed{PS' = a + e|x_1|}$$

$$\boxed{PS + PS' = 2a \quad \text{\} Length of major axis}}$$

Alternative Defination of Ellipse

The locus of a pt. which moves in such a way that the sum of its distance from 2 fixed pts is always constant and greater than the distance b/w 2 fixed pts.

Ques: Mohan is running around an elliptical path in such a way that on the track his sum of distance from 2 flag posts is always equal to 10 and the distance b/w the flagposts is 8. If flags are considered in horizontal line. Find the equation of elliptical track.

Sol: $2a = 10 \Rightarrow a = 5$

$2ae = 8$

$10e = 8 \Rightarrow e = 4/5$

$$\frac{x^2}{a^2} + \frac{y^2}{a^2(1-e^2)} = 0$$

$$\frac{x^2}{25} + \frac{y^2}{25\left(1-\frac{16}{25}\right)} = 1$$

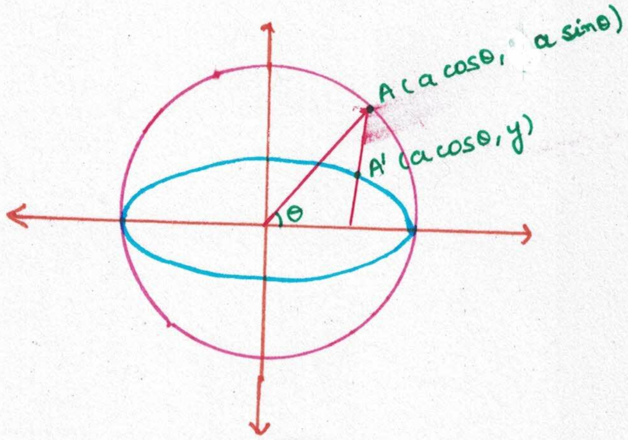
$$\frac{x^2}{25} + \frac{y^2}{9} = 1$$

Auxilary Circle

The locus of the foot of perpendicular drawn from focus on any tangent is known as auxillary circle.

In case of ellipse, $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, the equation of auxillary circle will be -

$$\boxed{x^2+y^2 = a^2}$$



$$\frac{a\cos^2\theta}{a^2} + \frac{y^2}{b^2} = 1$$

$$\frac{y^2}{b^2} = \sin^2\theta$$

$$y^2 = b^2\sin^2\theta$$

$$y = b\sin\theta$$

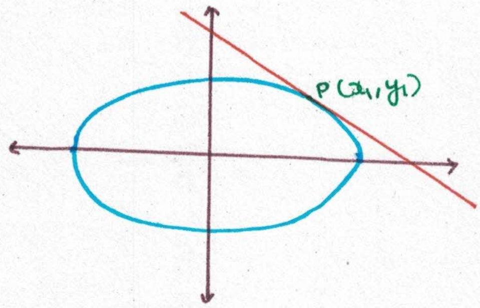
Parametric co-ordinates of ellipse = $(a\cos\theta, b\sin\theta)$

Here, θ is known as eccentric angle and $\theta \in [0, 2\pi]$

Equation of Tangent and Normal at a point

POINT FORM

$T=0$. If we wish to find the equation of a tangent at (x_1, y_1) to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, then it will be given as $T=0$



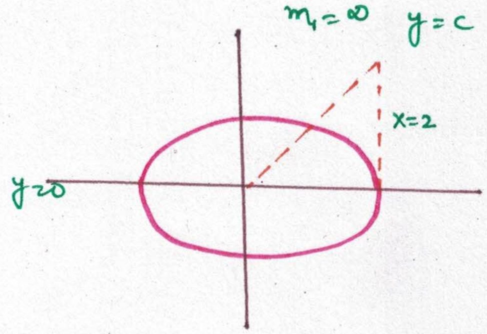
$$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} - 1 = 0$$

$$\boxed{\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1}$$

Ques: Find the equation of tangent and normal to the ellipse, $3x^2 + 4y^2 = 12$ at $(2,0)$ and also find the ratio of the Δ formed by these 2 lines with major axis with area of ellipse.

Sol:

$$\frac{x^2}{4} + \frac{y^2}{3} = 0 \quad , \quad a^2 = 4 \quad , \quad b^2 = 3$$



Tangent $-$ $\frac{x(2)}{4} = 1 \quad \Rightarrow x = 2$

$y=0$ normal

$$\text{area of ellipse} = \pi ab$$
$$= \pi \times 2 \times \sqrt{3}$$
$$= 2\sqrt{3}\,\pi$$

Equation of Normal

$$m_T = \frac{-x_1 b^2}{a^2 y_1}$$

$$m_N = \frac{a^2 y_1}{b^2 x_1}$$

Normal :- $(y - y_1) = \frac{a^2 y_1}{b^2 x_1}(x - x_1)$

$$\frac{(y-y_1)}{y_1} = \frac{a^2}{b^2}\left(\frac{x - x_1}{x_1}\right)$$

$$\frac{y}{y_1} - 1 = \frac{a^2}{b^2}\left(\frac{x}{x_1} - 1\right)$$

$$\frac{y}{y_1}b^2 - b^2 = \frac{a^2 x}{x_1} - a^2$$

$$\frac{a^2 x}{x_1} - \frac{b^2 y}{y_1} = a^2 - b^2$$

If ellipse is shifted, then $x = (x - \alpha)$, $y = (y - \beta)$

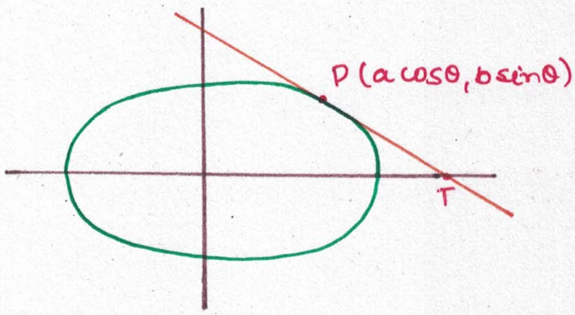
$x_1 = (x_1 - \alpha)$, $y_1 = (y_1 - \beta)$

PARAMETRIC FORM

Equation of tangent

$$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$$

$$\frac{x(a\cos\theta)}{a^2} + \frac{y(b\sin\theta)}{b^2} = 1$$



$$\boxed{\frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} = 1}$$

Equation of normal

$$\frac{a^2x}{a\cos\theta} - \frac{b^2y}{b\sin\theta} = a^2-b^2$$

$$\boxed{ax\sec\theta - by\,\text{cosec}\,\theta = a^2-b^2}$$

Ques: Find the equation of tangent whose eccentric angle is $\frac{\pi}{3}$ (60°) to the ellipse $\frac{x^2}{16} + \frac{y^2}{4} = 1$ and also find the POI of tangents when one pt. is given and another pt. is having difference of eccentric angle as $\frac{\pi}{2}$ in anticlockwise direction.

Sol: $a = 4$, $b = 2$

$$\frac{x}{2\times4} + \frac{y\sqrt{3}}{2\times2} = 1$$

$$\Rightarrow \frac{x + 2\sqrt{3}y}{8} = 1$$

$$\Rightarrow x + 2\sqrt{3}y = 8$$

$$\frac{\pi}{2} + \frac{\pi}{3} = \cos\left(\frac{\pi}{2} + \frac{\pi}{3}\right) = \sin\frac{\pi}{3} = -\frac{\sqrt{3}}{2}$$

$$\sin\theta = 1/2$$

$$\Rightarrow \frac{-x\sqrt{3}}{2\times4} + \frac{y}{2\times2} = 1$$

$$\Rightarrow \frac{-\sqrt{3}x + 2y}{8} = 1$$

$$\Rightarrow -\sqrt{3}(8 - 2\sqrt{3}y) + 2y = 8$$

$$\Rightarrow -8\sqrt{3} + 6y + 2y = 8$$

$$\Rightarrow 8\sqrt{3} - 4y = 8$$

$$\Rightarrow y = 2\sqrt{3} - 1$$

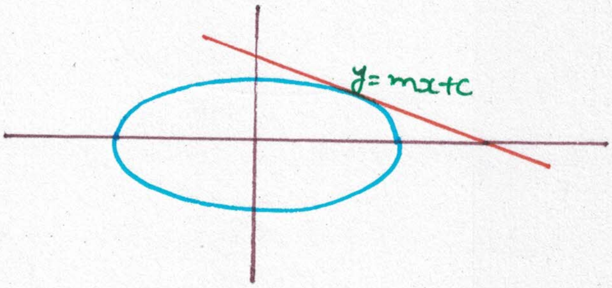
$$\Rightarrow x = 8 - 2\sqrt{3}(2\sqrt{3} - 1)$$

$$\Rightarrow x = 2\sqrt{3} - 4$$

SLOPE FORM

Let $y = mx+c$ is a tangent to ellipse

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$



$\Rightarrow \frac{x^2}{a^2} + \frac{m^2x^2+c^2+2mxc}{b^2} = 1$

$\Rightarrow b^2x^2 + a^2m^2x^2 + a^2c^2 + 2mxc\,a^2 = a^2b^2$

$\Rightarrow x^2(b^2+a^2m^2) + 2a^2mxc + a^2c^2 - a^2b^2 = 0$

$\Rightarrow D=0$

$\Rightarrow 4a^4m^2c^2 - 4(b^2+a^2m^2)(a^2c^2-a^2b^2) = 0$

$\Rightarrow a^4m^2c^2 - a^2b^2c^2 - a^2b^4 - a^4m^2c^2 - a^4m^2b^2 = 0$

$\Rightarrow a^4m^2b^2 + a^2b^4 - a^2b^2c^2 = 0$

$\Rightarrow a^2m^2 + b^2 - c^2 = 0$

$\Rightarrow a^2m^2 + b^2 = c^2$

$\Rightarrow c = \pm\sqrt{a^2m^2+b^2}$

$\Rightarrow y = mx+c$

$\Rightarrow \boxed{y = mx \pm \sqrt{a^2m^2+b^2}}$

Here, $\pm$ sign is used to obtain 2 parallel tangents with one slope.

POINT OF CONTACT

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$\Rightarrow \frac{x^2}{a^2} + \frac{m^2x^2 + a^2m^2 + b^2 \pm 2mx\sqrt{a^2m^2+b^2}}{b^2} = 1$

$\Rightarrow x^2b^2 + a^2m^2x^2 + a^4m^2 + a^2b^2 \pm 2a^2mx\sqrt{a^2m^2+b^2} = a^2b^2$

$\Rightarrow x^2(m^2a^2+b^2) \pm 2(x\sqrt{a^2m^2+b^2}) + a^4m^2 = 0$

$\Rightarrow (x\sqrt{a^2m^2+b^2} \pm a^2m) = 0$

$\Rightarrow x = \mp \frac{a^2m}{\sqrt{a^2m^2+b^2}}$

$y = m\left(\mp \frac{a^2m}{\sqrt{a^2m^2+b^2}}\right) \pm \sqrt{a^2m^2+b^2}$

$$y = \frac{\pm m^2a^2 \pm a^2m^2 \pm b^2}{\sqrt{a^2m^2+b^2}}$$

$$y = \frac{\pm b^2}{\sqrt{a^2m^2+b^2}}$$

$$\boxed{\text{Point of contact} = \left(\frac{\mp a^2 m}{\sqrt{a^2m^2+b^2}} , \pm \frac{b^2}{\sqrt{a^2m^2+b^2}} \right)}$$

EQUATION OF NORMAL IN SLOPE FORM

Point of contact of normal is same as POC of tangent $m_T \times m_n = -1$

$$\left(\frac{\mp a^2 m}{\sqrt{a^2m_T^2+b^2}} , \frac{\pm b^2}{\sqrt{a^2m_T^2+b^2}} \right)$$

$$\Rightarrow \left(\frac{\mp a^2(-1/m_n)}{\sqrt{a^2\left(\frac{1}{m_n^2}\right)+b^2}} , \frac{\pm b^2}{\sqrt{a^2+b^2m_n^2}} \right) = (x_1, y)$$

$$\Rightarrow \left(\frac{\pm a^2}{\sqrt{a^2+b^2m_n^2}} , \pm \frac{b^2 m_n}{\sqrt{a^2+b^2m_n^2}} \right) = (x, y)$$

$$\Rightarrow y - y_1 = m_n(x - x_1)$$

$$\Rightarrow y - \left(\frac{\pm b^2 m_n}{\sqrt{a^2+b^2m_n^2}} \right) = m_n \left(x - \frac{\pm a^2}{\sqrt{a^2+b^2m_n^2}} \right)$$

$$\boxed{(y - m_n x) = m_n \left(\frac{(b^2-a^2)}{\sqrt{a^2+b^2m_n^2}} \right)}$$

Q A

Ques: If $lx + my + n = 0$ is a normal to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, then prove that $\frac{n^2}{(a^2-b^2)^2}\left[\frac{a^2}{l^2} + \frac{b^2}{m^2}\right] = 1$

Sol: $-m_n x + y \pm m_n \left(\frac{b^2-a^2}{\sqrt{a^2+b^2m^2}} \right) = 0$

I) $\frac{l}{-m_n} = \frac{m}{1}$

$m_n = -\frac{l}{m}$

$$\frac{n}{\frac{-l}{m}\left(\frac{b^2-a^2}{\sqrt{a^2+b^2m^2}}\right)} = \frac{1}{l/m}$$

$$\frac{n^2m^2}{(a^2-b^2)^2 l^2}(a^2+b^2m^2) = \frac{1}{1/m^2}$$

$$\frac{n^2}{(a^2-b^2)}\left[\frac{a^2}{l^2}+\frac{b^2}{m^2}\right] = 1$$

II) $ax\sec\theta - by\,\mathrm{cosec}\,\theta = a^2-b^2$

$$\frac{ax\sec\theta}{a^2-b^2} - \frac{by\,\mathrm{cosec}\,\theta}{a^2-b^2} = 1$$

$$lx + my + n = 0$$

$$\frac{lx}{-n} + \frac{my}{-n} = 1$$

$$\frac{a\sec\theta}{a^2-b^2} = \frac{-l}{n} \qquad \frac{b\,\mathrm{cosec}\,\theta}{a^2-b^2} = -\frac{m}{n}$$

$$\cos\theta = -\frac{n}{l}\,\frac{a}{(a^2-b^2)} \qquad \sin\theta = \frac{n}{m}\,\frac{b}{(a^2-b^2)}$$

$$\cos^2\theta + \sin^2\theta = 1 = \frac{n^2a^2}{l^2(a^2-b^2)} + \frac{n^2b^2}{m^2(a^2-b^2)}$$

$$= \frac{n^2}{(a^2-b^2)}\left(\frac{a^2}{l^2}+\frac{b^2}{m^2}\right)$$

Ques: If $x\cos\alpha + y\sin\alpha = p$ touches the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, then prove that $a^2\cos^2\alpha + b^2\sin^2\alpha = p^2$

Sol:

$$\frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} = 1 \qquad \frac{x\cos\alpha}{p} + \frac{y\sin\alpha}{p} = 1$$

$$\frac{\cos\theta}{a} = \frac{\cos\alpha}{p} \qquad \frac{\sin\theta}{b} = \frac{\sin\alpha}{p}$$

$$a\cos\theta = \frac{a^2\cos\alpha}{p} \qquad b\sin\theta = \frac{b^2\sin\alpha}{p}$$

$$\frac{\left(\frac{a^2\cos\alpha}{p}\right)^2}{a^2} + \frac{\left(\frac{b^2\sin\alpha}{p}\right)^2}{b^2} = 1$$

$$a^2\cos^2\alpha + b^2\sin^2\alpha = p^2$$

Ques: If $lx + my + n = 0$ touches the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ then prove that $a^2l^2 + b^2m^2 = n^2$

Sol: $\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1$ $\qquad \frac{lx}{-n} + \frac{my}{-n} = 1$

$\frac{\cos\theta}{a} = \frac{l}{-n}$ $\qquad \frac{\sin\theta}{b} = \frac{m}{-n}$

$\Rightarrow \quad \cos^2\theta + \sin^2\theta = \frac{a^2l^2}{n^2} + \frac{b^2m^2}{n^2}$

$\Rightarrow \quad a^2l^2 + b^2m^2 = n^2$

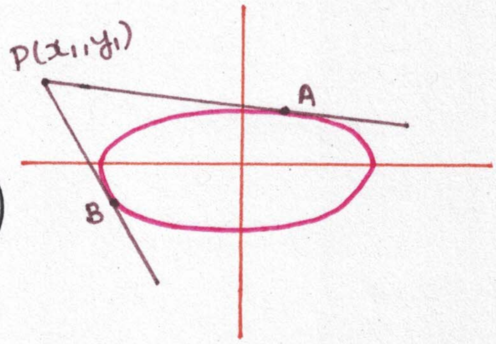
TANGENT FROM THE POINT

COMBINED FORM

Combined equation of PA and PB

$T^2 = SS_1$

$$\left(\frac{xx_1}{a^2} + \frac{yy_1}{b^2} - 1\right) = \left(\frac{x}{a^2} + \frac{y}{b^2}\right)\left(\frac{x_1}{a^2} + \frac{y_1}{b^2}\right)$$



SEPARATE FORM

If we wish to find the equation of tangents drawn from any pt. (x_1, y_1) which is lying outside the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, separately, we use following steps:

① Write the equation of tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ in slope form

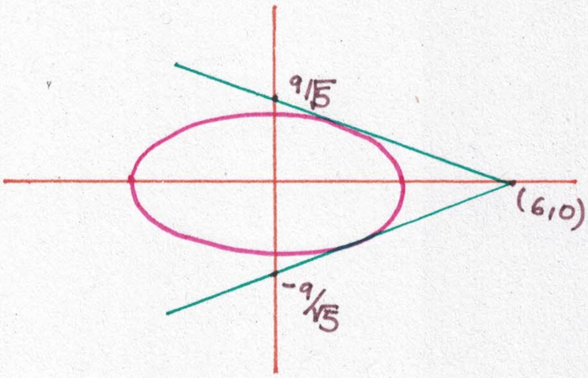
② Substitute the pt. $P(x_1, y_1)$ in the equation of tangent and make a quadratic equation in 'm'.

③ We will get 2 values of m and 4 equations of line out of which only 2 lines will pass from $P(x_1, y_1)$

NOTE

Sometimes, inspite of forming a quadratic equation in m, we will get a linear equation in m, It means one tangent will be vertical tangent passing through x, y_1 $\quad x = x_1$.

Ques: Find the eq^n of tangent drawn from the pt. (6,0) to the ellipse $\frac{x^2}{16}+\frac{y^2}{9}=1$ and also find the area of the Δ formed by these 2 tangents with y-axis.



Sol: $y_1 = mx_1 \pm \sqrt{a^2m^2+b^2}$

$0 = m(6) \pm \sqrt{16m^2+9}$

$36m^2 = 16m^2+9$

$20m^2 = 9$

$m = \frac{\pm 3}{2\sqrt{5}}$

$y = \frac{3}{2\sqrt{5}}x + \sqrt{16\left(\frac{9}{20}\right)+9}$

$y = \frac{3}{2\sqrt{5}}x + \frac{\sqrt{144+180}}{2\sqrt{5}}$

$y = \frac{3x+\sqrt{324}}{2\sqrt{5}}$

$y = \frac{3x+18}{2\sqrt{5}}$

$y = -\frac{3}{2\sqrt{5}}x + \sqrt{16\left(\frac{9}{20}\right)+9}$

$y = \frac{-3x+18}{2\sqrt{5}}$

$y = \frac{-3x+18}{2\sqrt{5}}$

$2\sqrt{5}y = 3x+18$ ✗

$2\sqrt{5}y = 3x-18$ ✓

$2\sqrt{5}y = -3x+18$ ✓

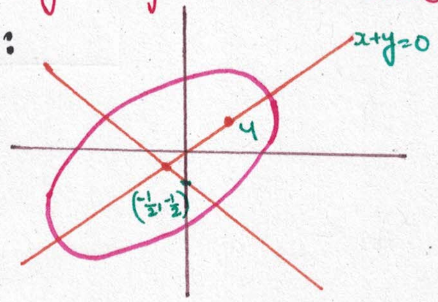
$2\sqrt{5}y = -3x-18$ ✗

area $= \frac{1}{2} \times 6 \times \frac{18}{\sqrt{5}}$

$= \frac{54}{\sqrt{5}}$

Ques: If the equation of major axis is $x-y=0$ and equation of minor axis is $x+y+1=0$. If length of major axis is 8 and length of minor axis is 4, then find the equation of ellipse.

Sol:



$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

$\frac{\left(x+\frac{1}{2}\right)^2}{16} + \left(y+\frac{1}{2}\right)^2$

$b = 2 \qquad a = 4$

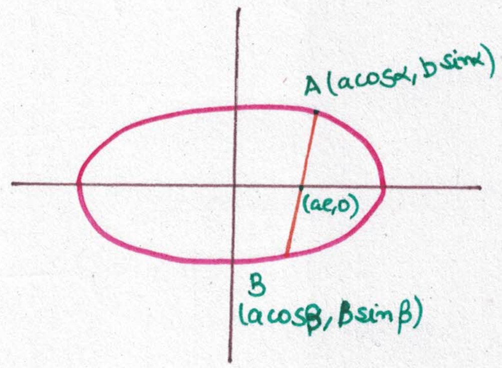
$$e = \sqrt{\frac{16-4}{16}} = \frac{\sqrt{12}}{4} = \frac{\sqrt{3}}{2}$$

$$\tan\theta = \pm 1 \qquad \cos\theta = -\frac{1}{\sqrt{2}} \qquad \sin\theta = \frac{1}{\sqrt{2}}$$

$$\frac{4}{\sqrt{2}}, \frac{4}{\sqrt{2}} = (2\sqrt{2}, 2\sqrt{2})$$

$$(x - 2\sqrt{2})^2 + (y - 2\sqrt{2})^2 = \frac{3}{4} \Big\} \text{ Equation of ellipse}$$

Ques: If a chord is drawn to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where the extremities are having eccentric angle α and β, then find the equation of chord as well as the value of $\tan\frac{\alpha}{2}, \tan\frac{\beta}{2}$ if this chord is a focal chord.



Sol:
$$AB = \frac{b(\sin\alpha - \sin\beta)}{a(\cos\alpha - \cos\beta)} = \frac{y - b\sin\alpha}{x - a\cos\alpha}$$

$$\Rightarrow y - b\sin^2\alpha = \frac{b}{a}\left(\frac{\sin\beta - \sin\alpha}{\cos\beta - \cos\alpha}\right)$$

$$\Rightarrow \frac{y}{b} - \sin^2\alpha = \frac{-2\cos\left(\frac{\beta+\alpha}{2}\right)\sin\left(\frac{\beta-\alpha}{2}\right)}{2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)}$$

$$\Rightarrow \frac{y}{b}\sin\alpha = \frac{-\cos\left(\frac{\beta+\alpha}{2}\right)}{\sin\left(\frac{\alpha+\beta}{2}\right)}(x - a\cos\alpha)$$

$$\Rightarrow \frac{y}{b}\sin\left(\frac{\alpha+\beta}{2}\right) - \sin^2\alpha \ \sin\left(\frac{\alpha+\beta}{2}\right) = -\frac{x}{a}\cos\left(\frac{\alpha+\beta}{2}\right) + \cos\alpha\cos\left(\frac{\alpha+\beta}{2}\right)$$

$$\Rightarrow \frac{x}{a}\cos\left(\frac{\alpha+\beta}{2}\right) + \frac{y}{b}\sin\left(\frac{\alpha+\beta}{2}\right) = \cos\alpha\cos\left(\frac{\alpha+\beta}{2}\right) + \sin\alpha\sin\left(\frac{\alpha+\beta}{2}\right)$$

$$\Rightarrow \boxed{\frac{x}{a}\cos\left(\frac{\alpha+\beta}{2}\right) + \frac{y}{b}\sin\left(\frac{\alpha+\beta}{2}\right) = \cos\left(\frac{\alpha-\beta}{2}\right)}$$

If this chord is focal chord, it pass through $(\pm ae, 0)$

$$\frac{ae}{a}\cos\left(\frac{\alpha+\beta}{2}\right) = \cos\left(\frac{\alpha-\beta}{2}\right)$$

$$\pm e = \frac{\cos\left(\frac{\alpha-\beta}{2}\right)}{\cos\left(\frac{\alpha+\beta}{2}\right)}$$

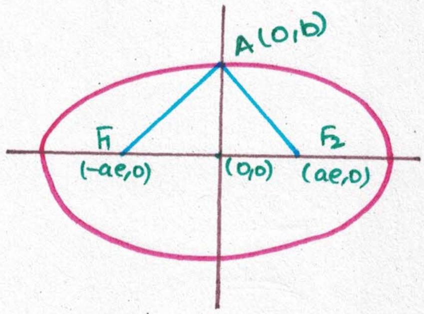
$$\frac{\pm e - 1}{\pm e + 1} = \cos\left(\frac{\alpha-\beta}{2}\right) - \cos\left(\frac{\alpha+\beta}{2}\right) \Big/ \cos\left(\frac{\alpha-\beta}{2}\right) + \cos\left(\frac{\alpha+\beta}{2}\right)$$

$$\frac{\pm}{\pm}\frac{e-1}{e+1} = \frac{2\sin(\alpha/2)\sin(\beta/2)}{2\cos(\alpha/2)\cos(\beta/2)} = \tan\frac{\alpha}{2}\cdot\tan\frac{\beta}{2}$$

Ques: If we join one end pt. of minor axis with focii of ellipse, $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and it makes right $\angle$ at minor axis then find the eccentricity of such ellipse.

Sol:

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$



$$m_1 m_2 = -1$$

$$\frac{b}{ae}\cdot\frac{b}{-ae} = -1$$

$$\frac{b^2}{a^2e^2} = 1 \Rightarrow e = b/a$$

$$\frac{\sqrt{a^2-b^2}}{a} = \frac{b}{a} \qquad\qquad \frac{\sqrt{b^2-a^2}}{a} = \frac{b}{a}$$

$$a^2-b^2 = b^2 \qquad\qquad b^2-a^2 = b^2$$

$$2b^2 = a^2 \qquad\qquad a=0$$

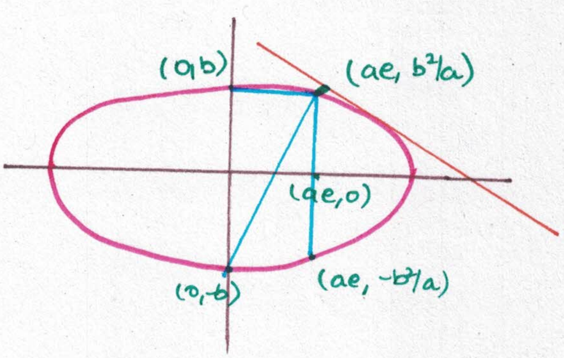
$$b = a/\sqrt{2}$$

$$e = \frac{a}{\sqrt{2}\,a} = \frac{1}{\sqrt{2}}$$

Ques: If in ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, a normal is drawn at the ext-remity of latus ractum and pass through one end of the minor axis then find the eccentricity of the ellipse as well as the value of $e^4+e^2-1=?$

Sol:

$$\frac{a^2x}{x_1} - \frac{b^2y}{y_1} = a^2-b^2$$



$$\frac{a^2x}{ae} - \frac{b^2y}{b^2/a} = a^2-b^2$$

it passes through $(0,-b)$

$$-\frac{b^2(-b)}{b^2} = a^2-b^2$$

$$ba = a^2-b^2$$

$$e^2 = 1-\frac{b^2}{a^2}$$

$$e^2 = 1+b/a-1$$

$e^2 = \frac{b}{a}$ $\qquad \frac{ab}{a^2} = \frac{a^2}{a^2} - \frac{b^2}{a^2}$

$e = \frac{\sqrt{ab}}{a} = \sqrt{b/a}$ $\qquad \frac{b}{a} = e$ $\qquad ab = (a^2 - b^2)$

$e^2 = \frac{b}{a}$, $e^4 = \frac{b^2}{a^2}$

$e^2 = 1 - \frac{b^2}{a^2}$ $\qquad e^2 = 1 - e^4$

$e^4 + e^2 - 1 = 0$

Let $e^2 = t$

$t^2 + t - 1 = 0$

$t = \frac{-1 \pm \sqrt{1+4}}{2}$ $\Rightarrow$ $e^2 = t = \frac{-1+\sqrt{5}}{2}$

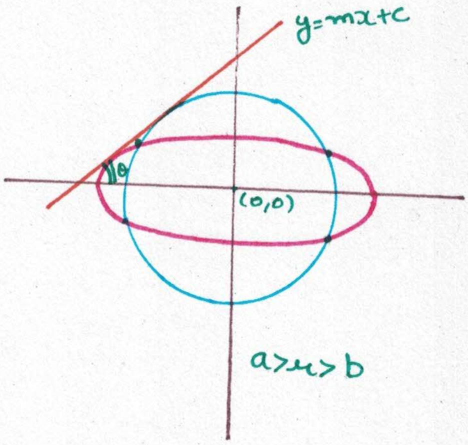
$e = \sqrt{\frac{\sqrt{5}-1}{2}}$

Ques: A circle $x^2+y^2=r^2$ is intersecting an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at 4 distinct pts. then find the angle of inclination of common tangent to both the curves.

Sol:

$y = mx \pm r\sqrt{1+m^2}$

$y = mx \pm \sqrt{a^2m^2+b^2}$



$r\sqrt{1+m^2} = \sqrt{a^2m^2+b^2}$

$r^2 + r^2m^2 = a^2m^2 + b^2$

$r^2 - b^2 = m^2(a^2 - r^2)$

$\frac{r^2-b^2}{a^2-r^2} = m^2$

$m = \sqrt{\frac{r^2-b^2}{a^2-r^2}}$

$\tan\theta = \pm\sqrt{\frac{r^2-b^2}{a^2-r^2}}$

$\theta = \tan^{-1}\left(\pm\sqrt{\frac{r^2-b^2}{a^2-r^2}}\right)$

Ques: From a pt. $P(x_1, y_1)$ two tangents are drawn to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, which are meeting the ellipse at $A(\alpha)$ and $B(\beta)$, then find the value of x_1, y_1 in terms of α, β

Sol:

Chord of contact – $y - b\sin\alpha = \dfrac{b(\sin\alpha - \sin\beta)}{a(\cos\alpha - \cos\beta)}(x - a\cos\alpha)$

$y(a\cos\alpha - \cos\beta) = x\,(b\sin\alpha - b\sin\beta)$

$\dfrac{xx_1}{a^2} + \dfrac{yy_1}{b^2} = 1$

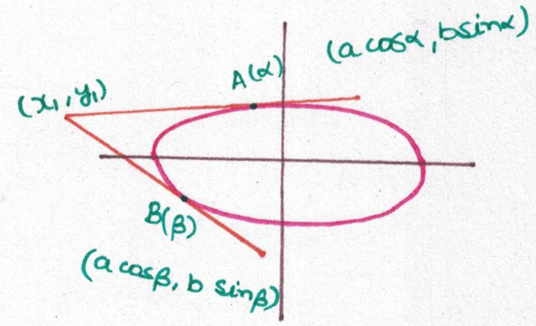
$$\frac{y_1}{b^2 a(\cos\alpha - \cos\beta)} = \frac{-x_1}{a^2 b(\sin\alpha - \sin\beta)} = \frac{1}{b\sin\alpha}$$

$$\frac{x}{a}\cos\left(\frac{\alpha+\beta}{2}\right) + \frac{y}{b}\sin\left(\frac{\alpha+\beta}{2}\right) = \cos\left(\frac{\alpha-\beta}{2}\right)$$

$$\frac{x_1}{a^2} = \frac{\cos\left(\frac{\alpha+\beta}{2}\right)}{a\cos\left(\frac{\alpha-\beta}{2}\right)}$$

$$x_1 = \frac{a\cos\left(\frac{\alpha+\beta}{2}\right)}{\cos\left(\frac{\alpha-\beta}{2}\right)} \qquad y_1 = \frac{b\sin\left(\frac{\alpha+\beta}{2}\right)}{\sin\left(\frac{\alpha-\beta}{2}\right)}$$

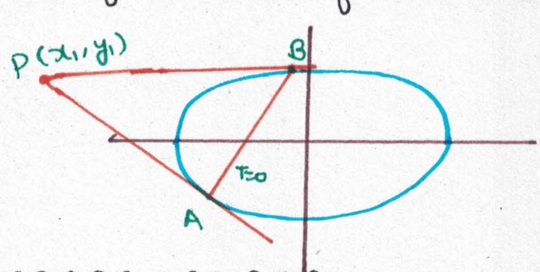
NOTE If we draw 2 tangents at the extremities of a chord whose end pts. are $A(\alpha)$, $B(\beta)$ then POI of tangents will be $\dfrac{a\cos\left(\frac{\alpha+\beta}{2}\right)}{\cos\left(\frac{\alpha-\beta}{2}\right)}, \dfrac{b\sin\left(\frac{\alpha+\beta}{2}\right)}{\cos\left(\frac{\alpha-\beta}{2}\right)}$



EQUATION OF THE CHORD OF CONTACT

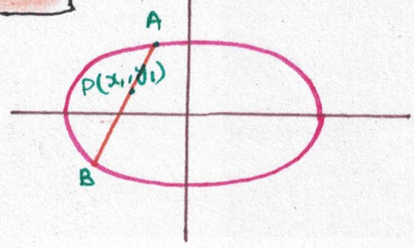
If we draw 2 tangents from some pt. $P(x_1, y_1)$ to the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ then the equation of chord of contact will be given as $T=0$.

$$\boxed{T = \frac{xx_1}{a^2} + \frac{yy_1}{b^2} - 1}$$



EQUATION OF THE CHORD WHOSE MID PT IS GIVEN

If the mid pt. of a chord is given as (x_1, y_1) then the equation of chord will be given as $T=0$

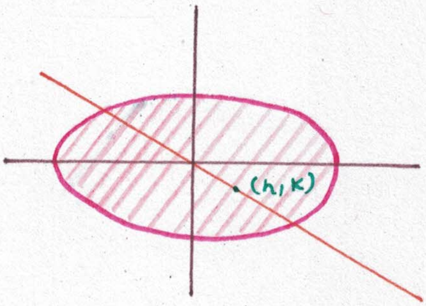


$$\boxed{\frac{xx_1}{a^2} + \frac{yy_1}{b^2} - 1 = \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} - 1}$$

Ques: Find the locus of mid pt. of parallel chords of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. (It will be diameter)

Sol: Eq. of chord AB $T = S_1$

$$\frac{hx}{a^2} + \frac{ky}{b^2} - 1 = \frac{h^2}{a^2} + \frac{k^2}{b^2} - 1$$



$$m = -\frac{hb^2}{a^2k}$$

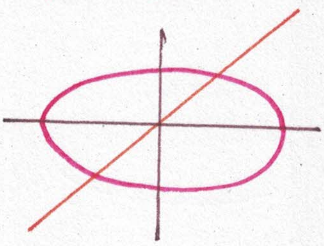
$$K = \frac{-hb^2}{a^2m}$$ $m \rightarrow$ slope of chord

$$\boxed{y = \frac{-b^2m}{a^2m}}$$ } Equation of diameter

Ques: Find the eq^n of a diameter of an ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$ and which is locus of all the chords whose slopes are 1

Sol: $y = \frac{-b^2x}{a^2m} = \frac{-9x}{16}$

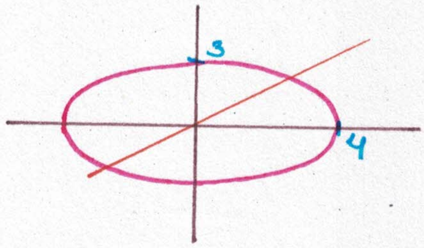
$$\boxed{16y + 9x = 0}$$



Ques: Find the eq^n of diameter of an ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$ which is passing through (3, 1) and this point is the mid pt. of a chord and diameter is making an angle 45° with it.

Sol: $y = mx$ $\quad 1 = m(3) \Rightarrow m = 1/3$

diameter $= \boxed{3y = x}$



DIRECTOR CIRCLE

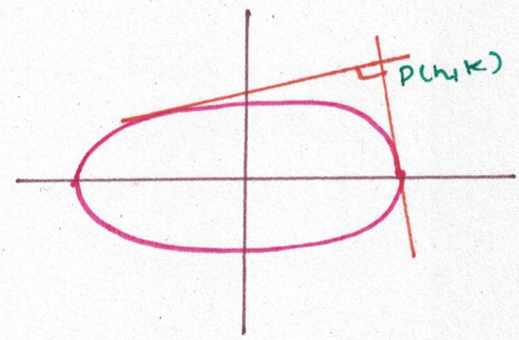
The locus of a point from which drawn tangents to ellipse are mutually perpendicular. It is given by the equation-

$$\boxed{x^2 + y^2 = a^2 + b^2}$$

Let the equation of tangent to ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ be -

$y = mx \pm \sqrt{a^2m^2 + b^2}$

It is passing through (h, k)



$k = mh \pm \sqrt{a^2m^2 + b^2}$

$(k - mh)^2 = \left(\sqrt{a^2m^2 + b^2}\right)^2$

$k^2 + m^2h^2 - 2khm = a^2m^2 + b^2$

$(h^2 - a^2)m^2 - (2kh)m + k^2 - b^2 = 0$

Let m_1, m_2 be the slopes of 2 tangent

then, $m_1 m_2 = -1$

$$\frac{k^2 - b^2}{h^2 - a^2} = -1$$

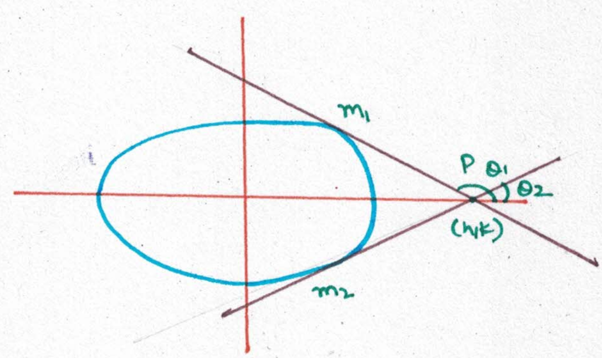
$k^2 - b^2 = a^2 - h^2$

$$\boxed{x^2 + y^2 = a^2 + b^2}$$

Ques: The locus of POI of 2 tangents to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ which are inclined at an angle θ_1 and θ_2 with major axis in such a way that $\cot\theta_1 + \cot\theta_2 = p$

Sol: $m_1 = \tan\theta_1$

$\cot\theta_1 = \frac{1}{m}$



$\frac{1}{m_1} + \frac{1}{m_2} = p$

$y = m_1x \pm \sqrt{a^2m_1^2 + b^2}$

$y = m_2x \pm \sqrt{a^2m_2^2 + b^2}$

$(h^2 - a^2)m^2 - (2kh)m + k^2 - b^2 = a$

$$\frac{2kh}{k^2 - b^2} = p$$

$2xy = p(y^2 - b^2)$

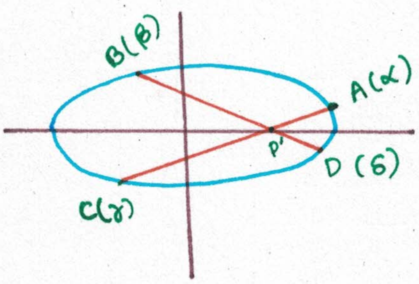
$p(y^2 - b^2) = 2xy$

$py^2 - 2xy = pb^2$

NOTE

1. From any point we can draw at max 4 normals to an ellipse.

A, B, C, D are co-normal pts.



$\alpha+\beta+\gamma+\delta$ = odd multiple of π

$\alpha, \beta, \gamma, \delta$ are eccentric angles of 4 conormal pts.

$$\tan(A+B+C+D) = \frac{S_1 - S_3}{1 - S_2 + S_4}$$

$$S_1 = \tan A + \tan B + \tan C + \tan D$$

$$\tan 3\theta = \frac{S_1 - S_3}{1 - S_2} \qquad \tan 5\theta = \frac{S_1 - S_3 + S_5}{1 - S_2 + S_4}$$

$$\sin(A+B+C) = \sin A \cos B \cos C + \sin B \cos A \cos C + \sin C \cos A \cos B - \sin A \sin B \sin B$$

$$= \frac{5\tan\theta - 10\tan^3\theta + \tan^5\theta}{1 - 10\tan^2\theta + 5\tan^4\theta}$$

* If $\alpha, \beta, \gamma, \delta$ are 4 pts. on an ellipse from which a circle is passing, then $\alpha+\beta+\gamma+\delta$ = even multiple of π

concyclic pts. $\rightarrow$ even

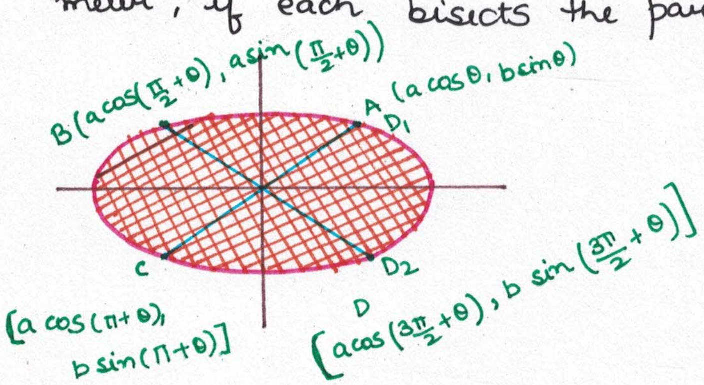
conormal pts. $\rightarrow$ odd

2. If $A(\alpha)$, $B(\beta)$, and $C(\gamma)$ are 3 pts. on ellipse from where the normals drawn are concurrent, then

$$\boxed{\sin(\alpha+\beta) + \sin(\beta+\gamma) + \sin(\alpha+\gamma) = 0}$$

3. The conormal points of an ellipse always lie on a fixed curve and these 4 pts. are POI of given ellipse and appolonus rectangular hyperbola

4. Two diameter of an ellipse are said to be conjugate diameter, if each bisects the parallel chords to other.



D_1, D_2 are conjugate diameter

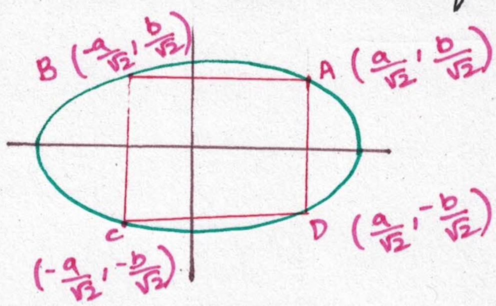
5. If $y = m_1 x$ and $y = m_2 x$ are two conjugate diameters, then $m_1 m_2 = -b^2/a^2$ and the extremity of the diameter will have

eccentric angle difference = $\pi/2$

Points to remember

1. If we draw tangents at the extremity of conjugate diameter, then they will form a quadrilateral whose area will be 4ab

2. Conjugate diameters are said to be equiconjugate diameter if θ is $\pi/4$.



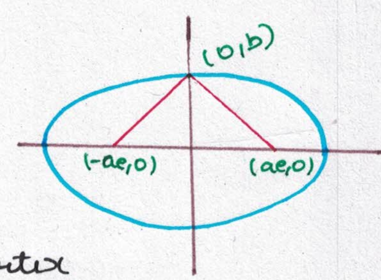
3. The greatest area of a rectangle inscribed in ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ will be 2ab.

4. The maximum area of a triangle inscribed in the ellipse in such a way that one vertex lies on ellipse and 2 other vertex are foci will be abe.

$$ar = \frac{1}{2}\begin{vmatrix} ae & 0 \\ -ae & 0 \\ a\cos\theta & b\sin\theta \\ ae & 0 \end{vmatrix} = \frac{1}{2}\left|-abe\sin\theta - abe\sin\theta\right|$$

$$= abe\sin\theta$$

$$(ar)_{max} = abe$$



5. The maximum are of the Δ formed by a movable pt. on the ellipse and vertex of the ellipse will be ab.

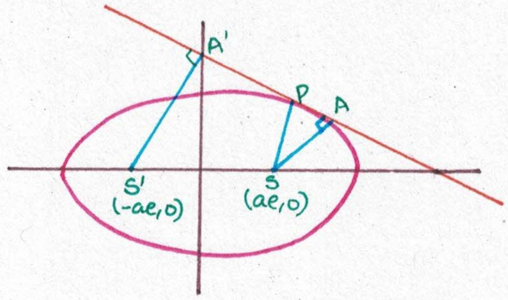
$$(ar)_{max} = ab$$

Properties of Ellipse

① All the diameters of ellipse will pass through the centre

② Foot of the perpendicular drawn from focus to any tangent to ellipse lies on auxillary circle

③ The product of the perpendiculars drawn from focus to any tangent of ellipe is always contant and equals to 'b^2'



$SA \cdot S'A' = b^2$

Tangent:

$$y = mx + \sqrt{a^2m^2+b^2}$$

$$\left|\frac{aem - 0 \pm \sqrt{a^2m^2+b^2}}{\sqrt{1+m^2}}\right| \quad \left|\frac{-aem - 0 \pm \sqrt{a^2m^2+b^2}}{\sqrt{1+m^2}}\right| \Big\} \ SA.S'A'$$

$$= \left|\frac{a^2m^2+b^2 - a^2e^2m^2}{1+m^2}\right|$$

$$= \left|\frac{a^2m^2(1-e^2)+b^2}{1+m^2}\right|$$

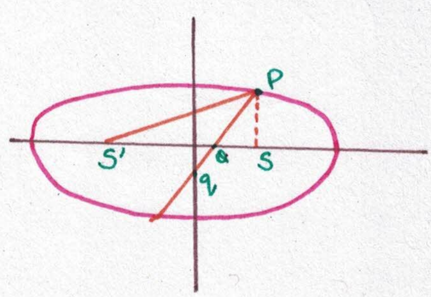
$$= \frac{m^2b^2+b^2}{1+m^2}$$

$$= \frac{b^2(1+m^2)}{1+m^2}$$

$$\boxed{SA \cdot S'A^2 = b^2}$$

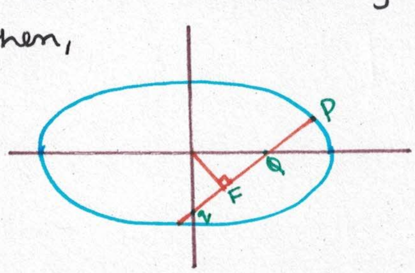
④ If we draw a normal at any point P which cuts the major axis at Q and minor axis at q. then,

$$\boxed{PQ \cdot Pq = SP \cdot S'P}$$



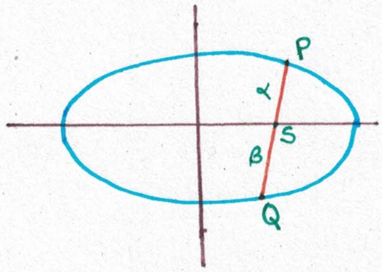
⑤ If we draw a normal at any pt. P to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ which cuts the major axis at Q and minor axis at q and the foot of the perpendicular drawn from center to this normal meets it at F, then,

$$\boxed{PF \cdot PQ = b^2}$$



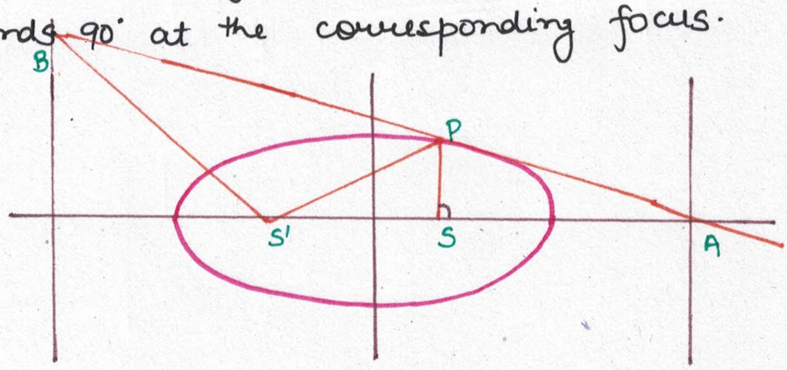
⑥ $PF \cdot Pq = a^2$

(7): If we draw a focal chord then the length of the segment of this focal chord between focus and ellipse are in such a way that harmonic mean of these distance = semi latus rectum

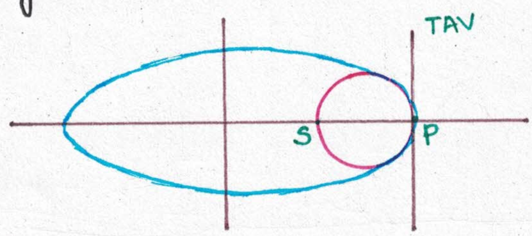


$$\boxed{\frac{2(PS)(SQ)}{PS+SQ} = \frac{b^2}{a}}$$

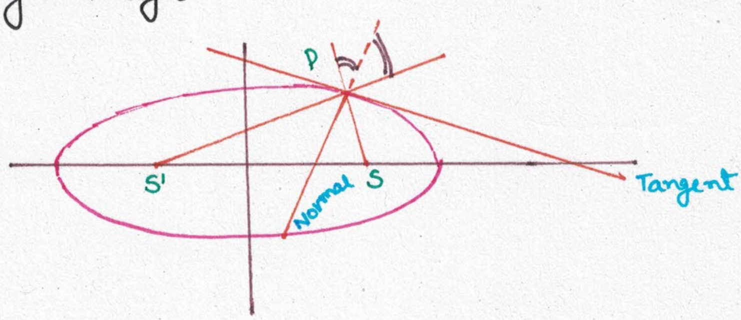
(8): The portion of the tangent between the point of contact and directrix subtends 90° at the corresponding focus.



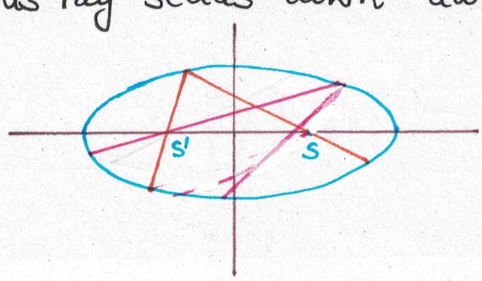
(9): If we take focal length as diameter of the circle, then this circle will always touch TAV



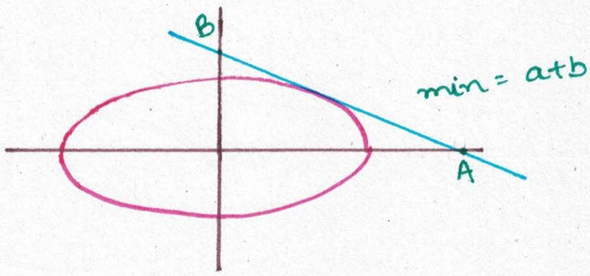
(10): The tangent and normal at any pt. P is an angle bisector of focal chord passing through



(11): If any ray passing through one focus of the ellipse then after reflection from ellipse it will pass through another focus and after infinite reflections, this ray settles down along the major axis.



(12) If we draw a tangent at any point P, then the distance of the portion b/w the co-ordinate axis is 'a+b' and this is minimum



(13) The locus of mid pt. of portion of normal between co-ordinate axis is another ellipse whose eccentricity = e of given ellipse.